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FARTHEST POINTS IN WEAKLY COMPACT SETS

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ABSTRACT
Let S be a weakly compact subset of a Banach space B. We show that the set of all points in B which have farthest points in S contains a dense G_δ of B. Also, we give a necessary and sufficient condition for bounded closed convex sets to be the closed convex hull of their farthest points in reflexive Banach spaces.

1. Introduction

Let B be a Banach space and let S be a bounded subset in B. We define a real valued function $r: B \to R$ by

$$r(x) = \sup\{\|x - z\| : z \in S\};$$

this is convex (it is the supremum of convex functions) and continuous, in fact, $|r(x) - r(y)| \leqq \|x - y\|$. A point $z \in S$ is called a *farthest point* of S if there exists an x in B such that $\|x - z\| = r(x)$. In [2], Edelstein showed that if B is a uniformly convex space and S is normed closed, then the set

$$D = \{x \in B : \|x - z\| = r(x) \quad \text{for some} \quad z \in S\}$$

is dense in B. The theorem was generalized by Asplund [1] to reflexive locally uniformly convex spaces; moreover, the set D was shown to contain a dense G_δ in B. In Section 2, we consider the subdifferential of the convex function r and, by a category argument, we can show that the theorem is true for any weakly compact subsets of a Banach space. In particular, our result implies Asplund's theorem.

In Section 3, we consider the Banach spaces B such that every bounded closed convex subset of B is the closed convex hull of its farthest points. A Banach space B is said to have *property* (I) if every bounded closed convex set in B is the intersection of a family of closed balls of B [4], [5]; we show that, if B is reflexive, then the above two properties are equivalent.

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2. The main theorem

Let B be a Banach space and let S be a bounded subset of B. For each $x \in B$, we define the *subdifferential* of the convex function r at x by

$$\partial r(x) = \{x^* \in B^* : \langle x^*, y - x\rangle + r(x) \leqq r(y) \quad \text{for all} \quad y \in B\}.$$

LEMMA 2.1. *Let B be a Banach space and let S be a bounded subset in B. Then for $x \in B$, each element of $\partial r(x)$ has norm less than or equal to 1.*

PROOF. For each $x \in B$, $x^* \in \partial r(x)$, we have

$$\langle x^*, y - x\rangle + r(x) \leqq r(y) \quad \text{for all} \quad y \in B.$$

Hence

$$\langle x^*, \ y - x\rangle \leqq r(y) - r(x) \leqq \| y - x \| \quad \text{for all} \quad y \in B,$$

i.e. $\|x^*\| \leqq 1$.

It is clear from the lemma that, for any x in B, $x^* \in \partial r(x)$, we have

$$\inf_{z \in S}\langle x^*, z - x\rangle \geqq - r(x).$$

LEMMA 2.2. *Let B be a Banach space and let S be a bounded subset in B. Then the set*

$$F = \{x \in B : \inf_{z \in S} \langle x^*, z - x\rangle > - r(x) \quad \text{for some} \quad x^* \in \partial r(x)\}$$

is of first category in B.

PROOF. Let

$$F_n = \{x \in B : \inf_{z \in S}\langle x^*, z - x\rangle \geqq - r(x) + \frac{1}{n} \quad \text{for some} \quad x^* \in \partial r(x)\},$$

then $F = \bigcup_{n=1}^{\infty} F_n$. We will show that, for any n, (i) F_n is a closed subset of B, (ii) F_n has empty interior.

(i) Let $\{x_m\}_{m=1}^{\infty}$ be a sequence in F_n which converges to an x in B. For each m, choose $x_m^* \in \partial r(x_m)$ such that

$$\inf_{z \in S}\langle x_m^*, \ z - x_m\rangle \geqq - r(x_m) + \frac{1}{n}.$$

Since $\|x_m^*\| \leqq 1$ for all m (Lemma 2.1), without loss of generality, we assume that $\{x_m^*\}_{m=1}^{\infty}$ converges weak* to x^*. We have, for any $y \in B$,

$$\begin{aligned}
&|\langle x_m^*, y - x_m\rangle - \langle x^*, y - x\rangle| \\
&\quad \leqq |\langle x_m^*, y - x_m\rangle - \langle x_m^*, y - x\rangle| + |\langle x_m^*, y - x\rangle - \langle x^*, y - x\rangle| \\
&\quad \leqq \|x_m - x\| + |\langle x_m^* - x^*, y - x\rangle|.
\end{aligned}$$

This shows that $\{\langle x_m^*, y - x_m\rangle\}_{m=1}^{\infty}$ converges to $\langle x^*, y - x\rangle$. Since $x_m^* \in \partial r(x_m)$,

$$\langle x_m^*, y - x_m\rangle + r(x_m) \leqq r(y) \quad \text{for all} \quad y \in B,$$

hence it follows that

$$\langle x^*, y - x\rangle + r(x) \leqq r(y) \quad \text{for all} \quad y \in B,$$

i.e., $x^* \in \partial r(x)$. Moreover,

$$\langle x_m^*, z - x_m\rangle \geqq -r(x_m) + \frac{1}{n} \quad \text{for all} \quad z \in S,$$

implies that

$$\langle x^*, z - x\rangle \geqq -r(x) + \frac{1}{n} \quad \text{for all} \quad z \in S,$$

i.e., $x \in F_n$ and F_n is a closed subset of B.

(ii) Suppose that some F_k has nonempty interior; then there exists an open ball U in B of radius 2λ and center at y_0 such that $U \subseteq F_k$. Let $\varepsilon = \lambda/4(1+\lambda)k$ and choose $z_0 \in S$ such that

$$r(y_0) - \varepsilon \leqq \|y_0 - z_0\| (\leqq r(y_0)).$$

Let

$$x_0 = y_0 + \lambda(y_0 - z_0).$$

Choose $x_1 \in U \subseteq F_k$ such that $\|x_1 - x_0\| < \varepsilon$. Then there exists $x_1^* \in \partial r(x_1)$ such that

$$\inf_{z \in S} \langle x_1^*, z - x_1\rangle \geqq -r(x_1) + \frac{1}{k}.$$

We shall show that

$$\langle x_1^*, y_0 - x_1\rangle + r(x_1) > r(y_0).$$

This will contradict the fact that x_1^* is a subdifferential of r at x_1 and complete the proof. Indeed,

$$\begin{aligned}
&r(y_0) - r(x_1) \\
&\quad < \left(\frac{1}{1+\lambda}\|x_0 - z_0\| + \varepsilon\right) - r(x_1) \\
&\quad < \left(\frac{1}{1+\lambda} r(x_1) + 2\varepsilon\right) - r(x_1)
\end{aligned}$$

$$= -\frac{\lambda}{1+\lambda} r(x_1) + 2\varepsilon$$

$$\leqq \frac{\lambda}{1+\lambda}\left(\langle x_1^*, z_0 - x_1\rangle - \frac{1}{k}\right) + 2\varepsilon$$

$$< \langle x_1^*, y_0 - x_1\rangle - \frac{\lambda}{(1+\lambda)k} + 4\varepsilon$$

$$= \langle x_1^*, y_0 - x_1\rangle .$$

THEOREM 2.3. *Let S be a weakly compact subset in a Banach space B. Then the set*

$$\{x \in B : \| x - z \| = r(x) \quad \text{for some} \quad z \in S\}$$

contains a dense G_δ of B. In particular, the set of farthest points of S is nonempty.

PROOF. Let F and F_n be defined as in Lemma 2.2 and let $D = B \setminus F$. Then

$$D = B \setminus \bigcup_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (B \setminus F_n),$$

where each $B \setminus F_n$ is an open, dense subset in B. Hence D is a dense G_δ in B. For each $x \in D$, $x^* \in \partial r(x)$, we have

$$\inf_{z \in S} \langle x^*, z - x\rangle = -r(x).$$

By weakly compactness of S, there exists a point $z_0 \in S$ with $\langle x^*, z_0 - x\rangle = -r(x)$. Hence

$$r(x) \geqq \| x - z_0 \| \geqq |\langle x^*, z_0 - x\rangle| = r(x).$$

This shows that $D \subseteq \{x : \| x - z \| = r(x) \text{ for some } z \in S\}$.

COROLLARY 2.4. *If B is a reflexive Banach space, then for every bounded, weakly closed subset in B, the set*

$$\{x \in B : \| x - z \| = r(x) \text{ for some } z \in S\}$$

contains a dense G_δ subset of B and hence the set of farthest points of S is nonempty.

COROLLARY 2.5 (Asplund). *Let B be a reflexive locally uniformly convex space, then Corollary 2.4 holds for every bounded, norm closed subset S in B.*

PROOF. By the locally uniformly convexity, each farthest point of $\overline{\text{conv}}\, S$ is a strongly exposed point of $\overline{\text{conv}}\, S$ and hence is contained in S. It follows that the sets of farthest points of S and $\overline{\text{conv}}\, S$ coincide. Hence we can apply Corollary 2.4 on $\overline{\text{conv}}\, S$.

3. Closed convex hulls of farthest points

In this section, we assume that S is a bounded closed convex subset of a Banach space. Let $b(S)$ denote the set of farthest points of S. Even in the two-dimensional spaces, the set S may fail to be the closed convex hull of its farthest points. (E.g., give R^2 the maximum norm and let $S = \{(x, y) : x^2 + y^2 \leqq 1\}$.)

A Banach space B is said to *have property* (I) if every bounded closed convex set in B can be represented as the intersection of a family of closed balls. This definition was introduced by Mazur [4] and was studied by Phelps [5]. The second author showed that there is a large class of Banach spaces (which includes those spaces whose duals are locally uniformly convex) with property (I). In [2], Edelstein proved that in a uniformly convex space with property (I), S is the closed convex hull of $b(S)$. However, the standing hypothesis that B is uniformly convex was used only to show that $b(S)$ is nonempty. Hence, by Theorem 2.3 and the proof of Theorem 2 in [2], we have

PROPOSITION 3.1 (Edelstein). *Suppose B is a Banach space with property (I); then every weakly compact convex subset of B is the closed convex hull of its farthest points.*

In the following, we shall prove the converse of the above proposition in the reflexive spaces.

LEMMA 3.2. *Let B be a Banach space. Suppose there exists a bounded closed convex subset S of B such that*

$$\bigcap \{C : C \text{ closed ball containing } S\} \supsetneqq S,$$

then there exists a bounded closed convex subset W with nonvoid interior such that

$$\bigcap \{C : C \text{ closed ball containing } W\} \supsetneqq W.$$

PROOF. Let

$$S_1 = \bigcap \{C : C \text{ closed ball containing } S\}.$$

Suppose $S_1 \supsetneqq S$, let $x_1 \in S_1 \setminus S$. By the separation theorem, we can find an $x^* \in B^*$ such that $\sup x^*(S) < x^*(x_1)$. Let W_0 be a bounded closed convex set with nonvoid interior and $\sup x^*(W_0) < \sup x^*(S)$. Let W be the closed convex hull of S and W_0, then $x_1 \notin W$ and it is clear that

$$x_1 \in S_1 \subseteq \bigcap \{C : C \text{ closed ball containing } W\}.$$

THEOREM 3.3. *Suppose B is a reflexive space; then B has property (I) if and only if every bounded closed convex subset in B is the closed convex hull of its farthest points.*

PROOF. The necessity follows from Proposition 3.1. To prove the sufficiency, let S be a bounded closed convex subset of B and let

$$S_1 = \bigcap \{C : C \text{ closed ball containing } S\}.$$

Suppose $S_1 \supsetneqq S$, there exists a point $x_1 \in S_1 \setminus S$. By the above lemma, we can assume that S has nonvoid interior; let y_1 be an interior point of S (hence an interior point of S_1) and choose z_1 such that

$$z_1 = \lambda x_1 + (1-\lambda) y_1,$$

with $0 < \lambda < 1$ and $z_1 \notin S$. Note that z_1 is then an interior point of S_1, so are any points of the form

$$(*) \qquad \alpha z_1 + (1-\alpha)x, \quad 0 < \alpha \leqq 1, \quad x \in S.$$

Let $S_2 = \operatorname{conv}(S \cup \{z_1\})$, we claim that $b(S_2)$, the set of farthest points of S_2, is contained in S. Indeed, for any $x \in B$, consider the function

$$r(x) = \sup\{\|x-y\| : y \in S\},$$

the ball $\{y \in B : \|x-y\| \leqq r(x)\}$ contains S and hence contains S_1 (by definition). Since each point of the form (*) is an interior point of S_1, its distance to x is less than $r(x)$ and cannot be a farthest point. It follows that $b(S_2) \subseteq S$, hence $z_1 \notin \overline{\operatorname{conv}}\, b(S_2)$; this contradicts that every bounded closed convex set in B is the closed convex hull of its farthest points, and the proof is complete.

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